Journal of Statistical Physics, Vol. 115, Nos. 3/4, May 2004 (© 2004)

On Asymptotics for the Airy Process

Harold Widom[1]

Received December 12, 2003; accepted December 24, 2003

The Airy process $t \to A(t)$, introduced by Prähofer and Spohn, is the limiting stationary process for a polynuclear growth model. Adler and van Moerbeke found a PDE in the variables s_1, s_2, and t for the probability $\Pr(A(0) \leqslant s_1, A(t) \leqslant s_2)$. Using this they were able, assuming the truth of a certain conjecture and appropriate uniformity, to obtain the first few terms of an asymptotic expansion for this probability as $t \to \infty$, with fixed s_1 and s_2. We shall show that the expansion can be obtained by using the Fredholm determinant representation for the probability. The main ingredients are formulas obtained by the author and C. A. Tracy in the derivation of the Painlevé II representation for the distribution function F_2 plus a few others obtained in the same way.

KEY WORDS: Airy process; asymptotics; Painlevé.

The Airy process $t \to A(t)$, introduced by Prähofer and Spohn[3] (see also ref. 2), is the limiting stationary process for a polynuclear growth model. It is also the limiting process for the largest eigenvalue of a Hermitian matrix whose entries undergo a Dyson Brownian motion. For any fixed t the probability $\Pr(A(t) \leqslant s)$ equals the distribution function $F_2(s)$ which was shown in ref. 4 to be representable in terms of a certain Painlevé II function.

In ref. 1 Adler and van Moerbeke found a PDE in the variables s_1, s_2, and t for the probability

$$\Pr(A(0) \leqslant s_1, A(t) \leqslant s_2).$$

[1] Department of Mathematics, University of California, Santa Cruz, California 95064; e-mail: widom@math.ucsc.edu

0022-4715/04/0500-1129/0

Using this they were able, assuming the truth of a certain conjecture and appropriate uniformity, to obtain the first few terms of an asymptotic expansion for this probability as $t \to \infty$, with fixed s_1 and s_2. It had the form

$$c_2(s_1, s_2)\, t^{-2} + c_4(s_1, s_2)\, t^{-4} + O(t^{-6}),$$

with explicitly computed coefficients c_2 and c_4. The form of the coefficients suggests that the expansion might be obtained by using the Fredholm determinant representation for the probability, and we shall show here that this is so. After a straightforward computation the main ingredients will be some of the formulas obtained in ref. 4 to derive the Painlevé II representatioin of F_2 plus a few others obtained in much the same way.

We write χ_i for the function $\chi(x-s_i)$. The probability in question is the determinant of I minus the operator with kernel

$$\begin{pmatrix} \chi_1(x) \int_0^\infty \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z)\, dz\, \chi_1(y) & \chi_1(x) \int_0^\infty e^{-zt}\, \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z)\, dz\, \chi_2(y) \\ -\chi_2(x) \int_{-\infty}^0 e^{zt}\, \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z)\, dz\, \chi_1(y) & \chi_2(x) \int_0^\infty \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z)\, dz\, \chi_2(y) \end{pmatrix}.$$

If we set

$$K(x, y) = \int_0^\infty \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z)\, dz,$$

then the above equals

$$\begin{pmatrix} \chi_1(x)\, K(x, y)\, \chi_1(y) & 0 \\ 0 & \chi_2(x)\, K(x, y)\, \chi_2(y) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \chi_1(x) \int_0^\infty e^{-zt}\, \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z)\, dz\, \chi_2(y) \\ -\chi_2(x) \int_{-\infty}^0 e^{zt}\, \mathrm{Ai}(x+z)\, \mathrm{Ai}(y+z)\, dz\, \chi_1(y) & 0 \end{pmatrix}.$$

The second operator has an asymptotic expansion as $t \to \infty$ which, by easy estimates, is valid in trace norm. The upper-right corner is

$$t^{-1}\chi_1 \operatorname{Ai} \otimes \operatorname{Ai} \chi_2 + t^{-2}(\chi_1 \operatorname{Ai}' \otimes \operatorname{Ai} \chi_2 + \chi_1 \operatorname{Ai} \otimes \operatorname{Ai}' \chi_2)$$
$$+ t^{-3}(\chi_1 \operatorname{Ai}'' \otimes \operatorname{Ai} \chi_2 + 2\, \chi_1 \operatorname{Ai}' \otimes \operatorname{Ai}' \chi_2 + \chi_1 \operatorname{Ai} \otimes \operatorname{Ai}'' \chi_2) + \cdots,$$

while the lower-left corner is

$$-t^{-1}\chi_2 \operatorname{Ai} \otimes \operatorname{Ai} \chi_1 + t^{-2}(\chi_2 \operatorname{Ai}' \otimes \operatorname{Ai} \chi_1 + \chi_2 \operatorname{Ai} \otimes \operatorname{Ai}' \chi_1)$$
$$- t^{-3}(\chi_2 \operatorname{Ai}'' \otimes \operatorname{Ai} \chi_1 + 2\, \chi_2 \operatorname{Ai}' \otimes \operatorname{Ai}' \chi_1 + \chi_2 \operatorname{Ai} \otimes \operatorname{Ai}'' \chi_1) + \cdots.$$

Here $f \otimes g$ denotes either the function $f(x)\, g(y)$ or the operator with this kernel.

We will factor out I minus the main operator

$$\begin{pmatrix} \chi_1(x)\, K(x, y)\, \chi_1(y) & 0 \\ 0 & \chi_2(x)\, K(x, y)\, \chi_2(y) \end{pmatrix},$$

which has determinant $F_2(s_1)\, F_2(s_2)$. For the resulting operator we use the notations from refs. 4 and 5,

$$(I - \chi K \chi)^{-1}\, \chi \operatorname{Ai} = Q, \quad (I - \chi K \chi)^{-1}\, \chi \operatorname{Ai}' = P, \quad (I - \chi K \chi)^{-1}\, \chi \operatorname{Ai}'' = Q_1.$$

Here χ could be χ_1 or χ_2 and we use the notations $Q(s_1)$, $Q(s_2)$, etc. to distinguish them.[2]

After factoring out on the left I minus the main operator we are left with I minus the operator with matrix kernel T whose diagonal entries are zero, whose upper-right corner is

$$T_{12} := t^{-1} Q(s_1) \otimes \operatorname{Ai} \chi_2 + t^{-2}(P(s_1) \otimes \operatorname{Ai} \chi_2 + Q(s_1) \otimes \operatorname{Ai}' \chi_2)$$
$$+ t^{-3}(Q_1(s_1) \otimes \operatorname{Ai} \chi_2 + 2\, P(s_1) \otimes \operatorname{Ai}' \chi_2 + Q(s_1) \otimes \operatorname{Ai}'' \chi_2) + \cdots,$$

and whose lower-left corner is

$$T_{21} := -t^{-1}\, Q(s_2) \otimes \operatorname{Ai} \chi_1 + t^{-2}(P(s_2) \otimes \operatorname{Ai} \chi_1 + Q(s_2) \otimes \operatorname{Ai}' \chi_1)$$
$$- t^{-3}(Q_1(s_2) \otimes \operatorname{Ai} \chi_1 + 2\, P(s_2) \otimes \operatorname{Ai}' \chi_1 + Q(s_2) \otimes \operatorname{Ai}'' \chi_1) + \cdots.$$

We have

$$\det(I - T) = \det(I - T_{12} T_{21}) = e^{\operatorname{tr} \log(I - T_{12} T_{21})}$$
$$= -\operatorname{tr} T_{12} T_{21} + \tfrac{1}{2}((\operatorname{tr} T_{12} T_{21})^2 - \operatorname{tr}(T_{12} T_{21})^2) + \cdots.$$

[2] In refs. 4 and 5 we defined $Q(s) = (I - K\chi)^{-1} \operatorname{Ai}$, etc., but the functions agree on (s, ∞) so in the end that will not matter.

To evaluate traces of products we use the fact $(f \otimes g)(h \otimes k) = (g, h)\, f \otimes k$, whose trace is $(g, h)(f, k)$, and from refs. 4 and 5 the notations

$$u = (Q, A\,\chi), \qquad v = (P, \mathrm{Ai}\,\chi) = (Q, \mathrm{Ai}'\,\chi),$$
$$w = (P, \mathrm{Ai}'\,\chi), \qquad u_1 = (Q_1, \mathrm{Ai}\,\chi) = (Q, \mathrm{Ai}''\,\chi).$$

We find that

$$\begin{aligned} \mathrm{tr}\, T_{12}T_{21} &= -u(s_1)\, u(s_2)\, t^{-2} - [v(s_1)\, v(s_2) \\ &\qquad + 2u_1(s_1)\, u(s_2) - w(s_1)\, u(s_2) + \text{reversed}]\, t^{-4} + \cdots, \\ \mathrm{tr}\, (T_{12}T_{21})^2 &= u(s_1)^2\, u(s_2)^2\, t^{-4} + \cdots, \end{aligned}$$

where "reversed" denotes the same terms as before but with s_1 and s_2 interchanged.

It follows first that $c_2(s_1, s_2) = F_2(s_1)\, F_2(s_2)\, u(s_1)\, u(s_2)$, and since $u = F_2'/F_2$ (see below) we have $c_2(s_1, s_2) = F_2'(s_1)\, F_2'(s_2)$ in agreement with ref. 1. We see also that $c_4(s_1, s_2)$ equals $F_2(s_1)\, F_2(s_2)$ times

$$v(s_1)\, v(s_2) + (2\, u_1(s_1) - w(s_1))\, u(s_2) + \text{reversed}.$$

At this point we use formulas from refs. 4 and 5, which give representations for u, v, w, and u_1 in terms of the function q which satisfies the P_{II} equation $q'' = sq + 2q^3$ and $q(s) \sim \mathrm{Ai}(s)$ as $s \to \infty$. (About half of these already appear in ref. 4.)

In the notation of formulas (2.15)–(2.18) of ref. 5, we have[3]

$$u' = -q^2, \qquad v' = -pq, \qquad w' = -p^2, \qquad u_1' = -q_1\, q,$$

and by formula (2.12) of ref. 5 with x replaced by s

$$q_1(s) = sq - vq + up.$$

By the formula for u', since $u(+\infty) = 0$,

$$u = \int_s^\infty q(x)^2.$$

(Thus $F_2'/F_2 = u$, as stated above.) Eventually everything will be expressed in terms of q and u.

[3] The definitions of the terms appearing on the right sides are $q = Q(s+)$, $p = P(s+)$, $q_1 = Q_1(s+)$.

We have $v' = -pq$ and formula (3.1) of ref. 4, which says $q' = p - qu$, gives

$$v' = -(q' + qu)\, q = -\tfrac{1}{2}\,(q^2)' + \tfrac{1}{2}\,(u^2)',$$

so

$$v = -\tfrac{1}{2}\,q^2 + \tfrac{1}{2}\,u^2.$$

Next,

$$u_1' = -q_1 q = -(sq - vq + up)\, q = -sq^2 + vq^2 - qu(q' + qu)$$

and

$$-w' = (q' + qu)^2,$$

which give

$$(2\,u_1 - w)' = -2sq^2 + 2vq^2 + (q')^2 - q^2u^2.$$

By the formula for v this is

$$-2sq^2 - q^4 + (q')^2,$$

and so

$$2\,u_1 - w = \int_s^\infty (2x\, q(x)^2 + q(x)^4 - q'(x)^2)\, dx.$$

Putting these together we find that $c_4(s_1, s_2)$ equals $F_2(s_1)\, F_2(s_2)$ times

$$\tfrac{1}{4}\, u(s_1)^2\, u(s_2)^2 + \tfrac{1}{4}\, q(s_1)^2\, q(s_2)^2 - \tfrac{1}{2}\, q(s_1)^2\, u(s_2)^2$$
$$+ \int_{s_1}^\infty (2x\, q(x)^2 + q(x)^4 - q'(x)^2)\, dx \cdot u(s_2) + \text{reversed}.$$

This looks a little different from the formula of ref. 1, because there the last integral is (we change s_1 back to s)

$$\int_s^\infty (2(s-x)\, q(x)^2 - q(x)^4 + q'(x)^2)\, dx.$$

But if we take d^2/ds^2 of the two integrals we see, using the equation $q'' = sq + 2q^3$ satisfied by q, that the results are the same. Since the integrals and their derivatives both vanish at $s = +\infty$ the integrals are equal.

ACKNOWLEDGMENTS

Supported by National Science Foundation Grant DMS-0243982.

REFERENCES

1. M. Adler and P. van Moerbeke, A PDE for the joint distribution of the Airy process, arXiv: math.PR/0302329.
2. K. Johansson, Discrete polynuclear growth and determinantal processes, arXiv: math.PR/0206208.
3. M. Prähofer and H. Spohn, Scale invariance of the PNG droplet and the Airy process, *J. Stat. Phys.* **108**:1071–1106 (2002).
4. C. A. Tracy and H. Widom, Level-spacing distributions and the Airy kernel, *Comm. Math. Phys.* **159**:151–174 (1994).
5. C. A. Tracy and H. Widom, Fredholm determinants, differential equations, and matrix models, *Comm. Math. Phys.* **163**:38–72 (1994).